ROBERT C. YATES

ANALYTIC GEOMETRY

with CALCULUS

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Preface

In recent years analytic geometry and the calculus have been combined into one course for the first or second year of college mathematics, and several excellent texts have been published for this purpose. However, these texts give primary emphasis to the calculus with a correspondingly reduced content in analytic geometry. As a consequence, many students are not acquiring a sound knowledge of analytic facts and techniques. For this reason, this book emphasizes a full-bodied treatment of analytic geometry in which the fundamental principles of calculus are introduced and used in a supporting role. The result is a text, to follow algebra and trigonometry, in which the student is more adequately prepared for the subject matter of calculus.

Certain features of the book are listed here as a guide to the reader before he begins a detailed study of the text. In the geometry of the plane, rectangular and polar coordinates are introduced at the same time and used interchangeably throughout. Lines are characterized by direction numbers for wider application and to facilitate transition to three-space geometry. The concepts of derivative and its inverse are presented early so that their applications to direction of a curve, motion of points, plane areas, tangents and normals to surfaces and curves in three-space, and volumes bounded by surfaces considerably enhance the subject matter. Discussion of the conics begins with the fundamental consideration of plane sections of a right circular cone, thus establishing their proper designation as conic sections. The customary 'higher plane' curves make their natural appearance as important loci connected with various mechanical devices such as cams, gears, and linkages. There are treatments of diameters of the conics and diametral planes of the quadrics which provide a firm foundation for further study of geometry. The usual treatment of transformations is amplified and extended to plane mappings, some given in matrix form. Ruled surfaces are studied and attention is focused upon their important application in the construction of space gears.

This book contains ample material for a full semester course of four hours a week or for a trimester of five hours a week. For a shorter course of three semester hours, certain sections (indeed, whole chapters) marked with stars may be omitted without discontinuity. Some of these sections, however, contain material that may well stimulate interest and should not be omitted without considering the student's future.

The book has been written for the student. It is supposed that this is his first introduction to analytic geometry, and to calculus. However, it is assumed that he has some knowledge of algebra (with determinants) and of trigonometry. The concepts of limit and derivative are presented in a manner to give the student clear comprehension and understanding. The formalized 'epsilon-delta' language is left to a later period when the student will be more mature and capable of better appreciation. However, half-truths and twilight meanings have been avoided.

The subject is to be enjoyed. It is often in this material that students first realize the fascination and compelling absorption of mathematics.

The author gratefully acknowledges the invaluable help of Mrs. Al Ferguson and Miss Carolyn Washer in preparing the typed manuscript, the cooperation of students at the College of William and Mary for their constructive and uninhibited criticism of several versions of trial texts, and the meticulous preparation of line drawings by his son, Daniel S. Yates.

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Tampa, Florida

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I.

THE PLANE

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1.1. The Real Number System and the Continuum Axiom

The counting numbers 1, 2, 3, $\cdots$ are called *natural numbers.* A basic assumption here is that *for every such number n there is a successor* $n + 1$. There is, accordingly, no last natural number. Together with 0 these numbers and their negatives comprise the *set of integers.*

A rational number is one that can be expressed as the quotient of two integers, viz., $\frac{2}{3}$, $\frac{7}{11}$, 6, 1, $-\frac{3}{5}$, 1.25, 0.454545 $\cdots$.

An irrational number is a real number that is not rational; i.e., one that cannot be expressed as the quotient of two integers, viz., $\sqrt{2}$, π, $\sqrt[3]{-7}$.

The set of all rational numbers (which includes the integers) and the irrational numbers form the *set of real numbers.* This set may be displayed as points on a line in the following fashion:

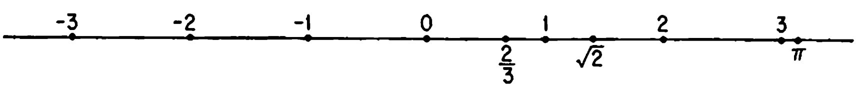

Fig. 1.1

From any selected zero point 0 we mark regularly spaced points to the right and left and label them by the *integers* in order of magnitude. The *rational numbers* may then be assigned to intermediate points and the *irrational numbers* to still other points.

The *continuum axiom* is the assumption that all real numbers may be paired uniquely with points on a line and conversely; i.e., each point of a line has a representative number. With points so labeled their order of magnitude has geometric meaning. Thus the pair of statements

$$3 < 5 \quad \text{and} \quad -7 < -2$$

means the point 5 lies to the right of 3, and -7 lies to the left of -2.

A further interpretation is possible and necessary. We wish to think of a real number not only as representing a point on a line but also as the signed *length* of the line segment to that point from the origin (the zero point). Segments are thus *directed.*

1.2. Coordinate Systems in the Plane

(A) *Rectangular Coordinate System.* We shall assume a plane upon which we select two perpendicular lines. Upon these lines, called x- and y-axes, directed segments measured from their intersection 0 (the origin) are taken to represent pairs of real numbers. We choose equal uniform scales on both axes. Such pairs (a, b) of ordered numbers locate points P, and conversely. That is, to every point P belongs a unique pair of numbers and to every pair of numbers there corresponds a point P. This is a *one-to-one correspondence.* These number pairs are called *coordinates* of P: the x-coordinate is the *abscissa,* the y-coordinate is the *ordinate.* It is to be understood that the first number in a pair represents a segment length in the x-direction, the second in the y-direction.

The selected lines taken as coordinate axes separate the plane into four regions called *quadrants,* labeled as shown. Signs of number pairs in these quadrants are, respectively $(+, +)$ $(-, +)$ $(-, -)$ $(+, -)$.

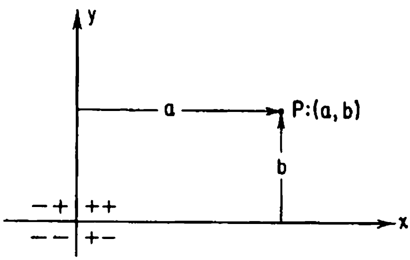

Fig. 1.2

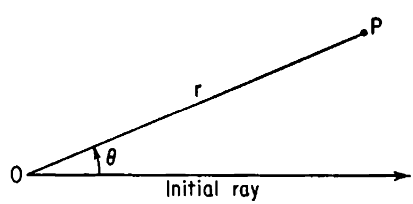

Fig. 1.3

(B) *Polar Coordinate System.* A second serviceable system in our assumed plane has for reference an *initial ray,* or 'half' line, and its initial end point 0, called the *pole.* An ordered pair of numbers (r, θ) then locates a point P whose directed distance from 0 is r, the *radius vector* of P; and this vector makes the angle θ with the initial ray. The angle of the pair is positive when measured from the initial ray in the *counterclockwise* direction, negative when clockwise. A negative distance r is to be interpreted as the extension of the radius vector "backward" through the pole 0. For example, $(-2, 30°)$ is plotted by drawing a radius vector at $+30°$ from the initial ray, then extending this line backward from O a length two units. It should be noted that although a single point is determined by a given pair of polar coordinates, the converse is not true. A selected point has an unlimited number of coordinate pairs. For example, the pairs $(2, 30°)$, $(-2, 210°)$, $(-2, -150°)$, and $(2, -330°)$ all designate the same point.

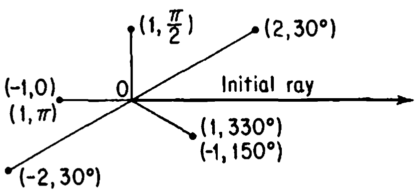

Fig. 1.4

If the angle is given in radian measure, there will be no symbol attached, thus $(2, \pi/6)$, $(2, 1.5)$.

(C) *Exchange of Systems.* The rectangular and polar coordinate systems may be exchanged one for the other by making the pole and the origin coincident, and the x-axis coincident with the initial ray as shown. Thus a point P may have coordinates (x, y) in the rectangular system and (r, θ) in the polar system. Relationships between these two sets of coordinates are apparently

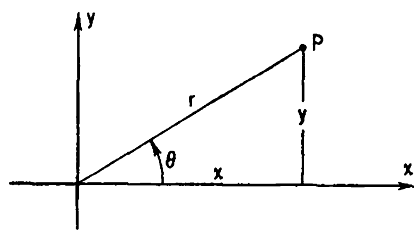

Fig. 1.5

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x). \end{cases}$$

These relationships permit the transfer of coordinates in one system to the other. For example, $(\sqrt{3}, 1)$ in rectangular coordinates can be written as $(2, 30°)$ in polar coordinates; whereas $(-1, 5\pi/6)$ polar, is $(\sqrt{3}/2, -\frac{1}{2})$ rectangular.

1.3. Graphs: Mathematical Statements

Equations or inequalities which relate x and y or r and θ define loci in the reference plane. That is, *the set of those points (and only those) whose* *coordinates satisfy such a statement forms the graphical representation or locus of the mathematical statement.**

For example, $3x + 2y = 12$ represents a straight line: the set of all points whose coordinates (x, y) satisfy the equation. All such pairs of numbers taken from the domain of all real numbers represent points on the line shown. Note that the line intercepts lengths of four and six units on the x- and y-axes, respectively.

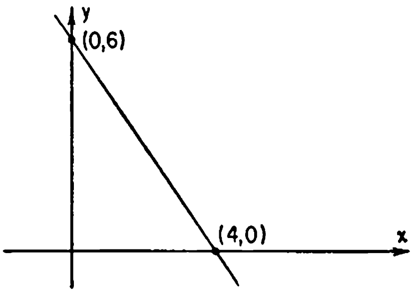

Fig. 1.6

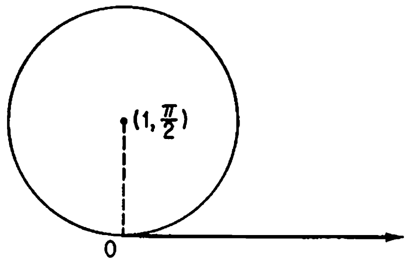

Fig. 1.7

As another example, $r = 2 \sin \theta$ represents a circle. A table of coordinates satisfying the equation is given, and the set of all such points forms the circle shown. This circle passes through the pole (since for $\theta = 0$, $r = 0$) and has its center at $(1, \pi/2)$. Such matters will be discussed fully in later sections.

Let us determine the rectangular equation of this circle. It is convenient (and legitimate) to multiply first by r

$$r^2 = 2r \sin \theta$$

before replacing these polar coordinates by their rectangular equivalents. Since $r^2 = x^2 + y^2$ and $r \sin \theta = y$, we have at once

$$x^2 + y^2 = 2y.$$

We will have occasion to determine points of the plane restricted by conditions of the sort

$$\{y \leqq 1, \quad y \gtreqqless x^2\}.$$

The set A of points satisfying the first restriction, $y \leqq 1$, is the shaded region shown in the left-hand picture. For all points in this region x may be any real number, but y may not be greater than $+1$. The set B for which $y \gtreqqless x^2$ consists of those points *on* the parabola $y = x^2$ together with

* Numbers which *satisfy* a mathematical statement are those which make the statement true.

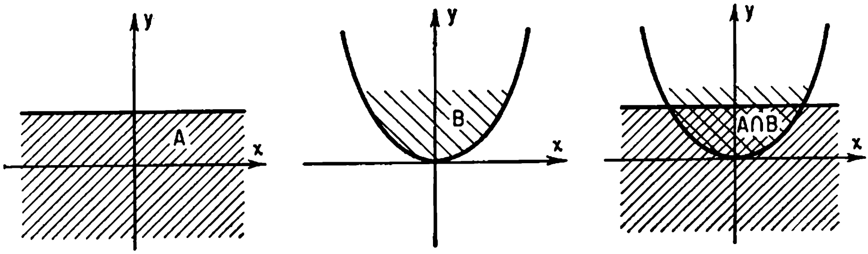

Fig. 1.8

all points 'above' the curve. The set of points common to the two sets (points whose coordinates satisfy both conditions) is called their *intersection* and denoted by the symbol $A \cap B$. This intersection set is shown as the cross-hatched region of the right-hand picture.

In working exercises pertaining to graphs, the student should list a table of number pairs for each equation. Chosen values of one variable should be listed in *order of magnitude* (from the domain of permissible values) and corresponding points then plotted. If it is not prohibited in some way, a smooth curve is then drawn through these points in order of their tabular arrangement. In doing this, it should be realized that an assumption has been made; i.e., points on the graph actually exist whose coordinates, if found, would satisfy the equation, and, of course, there are points on the curve only for *real* number pairs.

EXERCISES

1. Locate the points whose rectangular coordinates are

$(2, 3)$, $(-2, 3)$, $(-4, -2)$, $(0, 0)$, $(0, 3)$, $(3, 0)$, $(2, 1)$,

$(0, -2)$, $(-2, 0)$

2. Show that π radians $= 180°$

3. Convert to radian measure

$0°$, $90°$, $180°$, $270°$, $30°$, $45°$, $60°$, $120°$, $135°$, $150°$

$210°$, $225°$, $240°$, $315°$, $330°$, $-30°$, $-225°$

4. Convert to degree measure

$\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, π, $5\pi/4$, $-7\pi/6$, $-5\pi/4$,

$-7\pi/8$, 2, 3.5, -1, $-.75$

5. Sketch the region of the plane such that

(a) $\{x > 2, y > 1\}$

(b) $\{1 < x < 3, 1 < y < 2\}$

(c) $\{-1 < x < 0, -1 < y < 0\}$

(d) $\{x^2 < 4, y^2 < 9\}$

(e) $\{x \leqq 2, y \leqq 1\}$

(f) $\{ |x| \leqq 1, |y| \leqq 2\}$

6. Sketch the region of the plane such that

(a) $\{0 \leqq \theta \leqq \pi/6, 0 \leqq r \leqq 4\}$

(b) $\{0 \leqq \theta \leqq \pi, 1 \leqq r \leqq 2\}$

(c) $\{\pi/4 \leqq \theta \leqq 3\pi/4, |r| < 3\}$

(d) $\{\pi/4 \leqq \theta \leqq 3\pi/4, r \sin \theta < 3\}$

7. (a) Change the following polar coordinates to rectangular coordinates and plot the points

$(a, 0)$, $(0, a)$, $(1, \pi/2)$, $(-1, \pi/2)$, $(-1, -\pi/2)$, $(-2, 120°)$, $(-1, -\pi)$, $(-1, \pi)$

(b) Determine for each of the following points in rectangular coordinates four sets of polar coordinates with $0 \leqq \theta \leqq 2\pi$

$(0, 1)$, $(1, 0)$, $(-1, 0)$, $(0, -1)$, $(3, 1)$, $(1, -1)$

8. (a) Change the following statements to polar coordinates and sketch

$x + y = 3$, $2x - 3y < 1$, $y = 1$, $x = -2$, $x^2 + y^2 = y$

(b) Change the following statements to rectangular coordinates and sketch

$r \sin \theta = 1$, $r \cos \theta = -2$, $2r \sin \theta + 3r \cos \theta = 5$,

$r = 4$, $\theta = \pi/4$, $r = \sin \theta$, $r = 2 \cos \theta$, $0 < r - \cos \theta < 1$

1.4. Graphs, Rectangular (General Remarks)

In drawing graphs in rectangular coordinates the student will be guided by the following remarks (additional details will be supplied in later chapters).

(1) The graph of

$$y = ax + b$$

is a *straight line*, and the location of two of its points determines its position. Note particularly the points (O, b) and for $a \neq 0$, $(-b/a, O)$ where the line crosses the axes. These lengths b and $-b/a$ are *intercepts*. The graph of $x = k$ is a line parallel to the y-axis; $y = k$ is a line parallel to the x-axis.

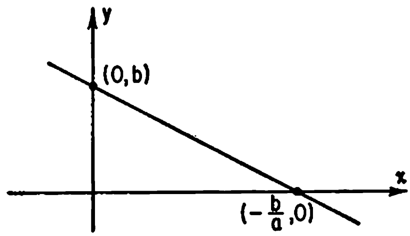

Fig. 1.9

(2) The graph of

$$y = ax^2 + bx + c, \qquad a \neq 0$$

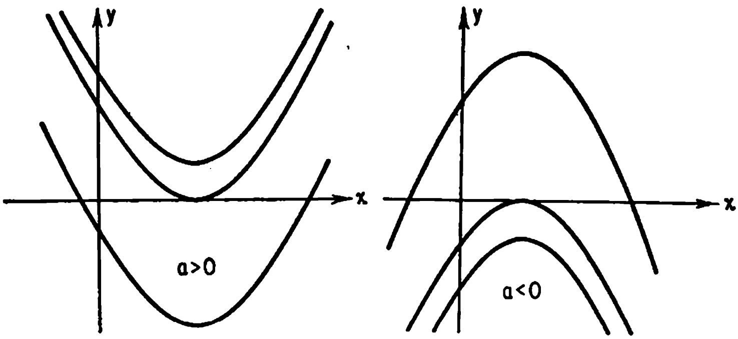

Fig. 1.10

is a parabola of the types shown. The domain of x is unrestricted, and for sufficiently large values of x, either positive or negative, the term ax^2 is numerically *dominant*. Thus if $a > 0$, the curve extends upward; if $a < 0$, it extends downward. The y-intercept is c; the x-intercepts (if any) are the real roots of $ax^2 + bx + c = 0$.

If we solve

$$ax^2 + bx + c - y = 0$$

for x:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac + 4ay}}{2a},$$

it is evident that lines y = constant (those parallel to the x-axis) intersect the curve in either two points or in no point depending upon the sign of the quantity $(b^2 - 4ac + 4ay)$. If this quantity is 0, however,

$$b^2 - 4ac + 4ay = 0;$$

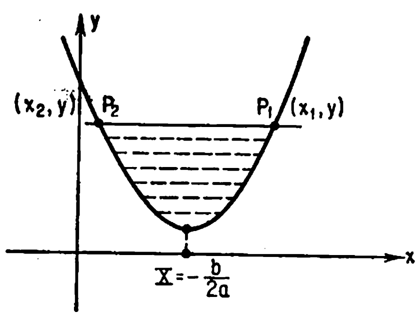

Fig. 1.11